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Generalized Noether Identities and Application to Yang–Mills Field Theory

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We derive generalized Noether identities for a system with noninvariant action integral under an infinite continuous group and deduce the string conservation laws of the system. We give a preliminary application to field theory and discuss the strong conservation laws for the BRS transformation and the weak conservation laws of Yang-Mills fields. The Dirac constraint of the system is examined.

1. INTRODUCTION

Noether's (1918) second theorem refers to invariance of the action integral under an infinite continuous group parametrized by r arbitrary functions and their derivatives. If an action integral is invariant under such a group, then there exist r differential identities (Noether identities), which involve the variational derivatives (Euler-Lagrange expressions). These identities are discussed by Hilbert (1924) and by Bergman and Anderson (1949, 1951) in connection with electrodynamics and general relativity, by Drobot and Rybarski (1958-1959) in connection with hydromechanics, by Sundermyer (1982) in connection with gauge field theory, and by others in a general way.

In the massive Yang-Mills (YM) theory the Lagrangian in general is not invariant under the gauge transformation; the gauge-invariant Lagrangian of Fermi fields and gauge fields is not invariant under the chirality transformation of the Fermi fields; the invariant Lagrangian under the BRS transformation is not invariant under the gauge transformation alone, etc. Therefore, we must discuss the transformation properties of systems that are not invariant under infinite continuous groups (Li, 1986).

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In this paper we discuss a more general case of a Lagrangian involving high-order derivatives. We derive generalized Noether identities (GNI) of the variant system, and using the GNI we obtain certain strong conservation laws. We give a preliminary application to field theory: the transformation of the gauge fields in connection with BRS invariance and strong conservation laws are discussed, and it is shown that for certain gauge-variant Lagrangians of massive YM fields along the trajectory of the motion the GNI can be converted to (weak) conservation laws. This differs from the usual result (first Noether theorem), where invariance under a finite continuous group implies the conservation laws. Finally, from the GNI we also examine Dirac's constraint of a gauge-variant system.

2. GENERALIZED NOETHER IDENTITIES

Suppose the system is described in terms of the state functions or field variables $\psi^\alpha(x)$, $\alpha = 1, 2, \ldots, n$. The Lagrangian density of the system, which may involve high-order derivatives of the state functions, is

$$\mathscr{L} = \mathscr{L}(x;\, \psi^\alpha(x),\, \psi^\alpha_{,\mu(m)}(x), \ldots) \tag{1}$$

where

$$\psi^\alpha_{,\mu(m)} = \partial_{\mu(m)}\psi^\alpha = \underbrace{(\partial_\mu\, \partial_\nu \cdots)}_{m}\psi^\alpha \tag{2}$$

Throughout the paper we adopt a Euclidean metric $x = (\boldsymbol{r}, it)$. The action integral is

$$I = \int_\Omega \mathscr{L}(x;\, \psi^\alpha,\, \psi^\alpha_{,\mu(m)}, \ldots)\, d\Omega \tag{3}$$

We consider the transformation properties of the system under the infinite continuous group whose infinitesimal transformation is

$$\begin{aligned} x_\mu \to x'_\mu &= x_\mu + R_{i\mu}\mathscr{E}^i \\ \psi^\alpha \to \psi^{\alpha'} &= \psi^\alpha + S^\alpha_i \mathscr{E}^i \end{aligned} \tag{4}$$

where $\mathscr{E}^i = \mathscr{E}^i(x)$, $i = 1, 2, \ldots, r$, are arbitrary independent functions, and $R_{i\mu}$ and S^α_i are linear differential operators:

$$\begin{aligned} R_{i\mu} &= a_{i\mu} + a_{i\mu\nu}\,\partial_\nu + \cdots + a_{i\mu\nu\cdots\rho}\,\partial_\nu \cdots \partial_\rho \\ S^\alpha_i &= b^\alpha_i + b^\alpha_{i\nu}\,\partial_\nu + \cdots + b^\alpha_{i\nu\cdots\rho}\,\partial_\nu \cdots \partial_\rho \end{aligned} \tag{5}$$

Under the transformation (4), suppose the change of the Lagrangian is

$$\delta\mathscr{L} = U_i \mathscr{E}^i \tag{6}$$

with

$$U_i = u_i^{\mu(m)}\,\partial_{\mu(m)} \tag{7}$$

where a, b, $u_i^{\mu(m)}$ are functions of the x, ψ^α, $\psi^\alpha_{,\mu(m)}$. The change of the action is (Li, 1985; Rosen, 1974)

$$\begin{aligned}\delta I &= \int_\Omega \frac{\delta \mathscr{L}}{\delta \psi^\alpha}(S_i^\alpha - \psi^\alpha_{,\mu} R_{i\mu})\mathscr{E}^i \, d\Omega \\ &\quad + \int_B d\sigma_\mu \left[\sum_{m=0}^{N-1} \Pi_\alpha^{\mu\nu(m)} \, \partial_{\nu(m)}(S_i^\alpha - \psi^\alpha_{,\rho} R_{i\rho})\mathscr{E}^i + \mathscr{L}\,\delta_{\mu\nu} R_{i\nu}\mathscr{E}^i\right] \\ &= \int_\Omega U_i \mathscr{E}^i \, d\Omega \end{aligned} \tag{8}$$

where

$$\frac{\delta \mathscr{L}}{\delta \psi^\alpha} = (-1)^m \, \partial_{\mu(m)} \mathscr{L}_\alpha^{\mu(m)} \tag{9}$$

$$\mathscr{L}_\alpha^{\mu(m)} = \frac{1}{m!} \sum_{\substack{\text{all permutation}\\ \text{of indices}}} \frac{\partial \mathscr{L}}{\partial \psi^\alpha_{,\mu(m)}} \tag{10}$$

$$\Pi_\alpha^{\mu\nu(m)} = \sum_{l=0}^{N-(m+1)} (-1)^l \, \partial_{\lambda(l)} \mathscr{L}_\alpha^{\mu\nu(m)\lambda(l)} \tag{11}$$

B is the boundary of domain Ω. Since $\mathscr{E}^i(x)$ are arbitrary, we may choose $\mathscr{E}^i(x)$ such that the boundary term in equation (8) vanishes, and repeat the integration by parts of the left-hand side of this identity. Again appealing to the arbitrariness of the $\mathscr{E}^i(x)$, we can force the boundary term to vanish, after which we can apply the fundamental lemma of the calculus of variations to conclude that

$$\tilde{S}_i^\alpha\left(\frac{\delta \mathscr{L}}{\delta \psi^\alpha}\right) - \tilde{R}_{i\mu}\left(\frac{\delta \mathscr{L}}{\delta \psi^\alpha}\psi^\alpha_{,\mu}\right) - \tilde{U}_i(1) = 0 \tag{12}$$

where the $\tilde{S}_i^\alpha$, $\tilde{R}_{i\mu}$, $\tilde{U}_i$ are the adjoint operators with respect to S_i^α, $R_{i\mu}$, U_i respectively, defined by

$$\int_\Omega g \tilde{S}_i^\alpha f \, d\Omega = \int_\Omega f S_i^\alpha g \, d\Omega + [\cdot]_B \tag{13}$$

where f, g are functions defined on domain Ω and $[\cdot]_B$ is a boundary term, and similar expressions hold for $R_{i\mu}$, $\tilde{R}_{i\mu}$ and U_i, $\tilde{U}_i$. In (12), $\tilde{U}_i(1)$ indicates the adjoint operator applied to unity. The expressions (12) are called the GNI (or generalized Bianchi identities) of the system with Lagrangian variant under the transformation (4).

Consider a special transformation

$$\begin{aligned} x_\mu &\to x'_\mu = x_\mu \\ \psi^\alpha &\to \psi^{\alpha'} = \psi^\alpha + a_i^\alpha \mathscr{E}^i + b_{i\mu}^\alpha \, \partial_\mu \mathscr{E}^i \end{aligned} \tag{14}$$

where $a_i^\alpha = a_i^\alpha(x; \psi^\alpha, \psi^\alpha_{,\mu})$ and $b_{i\mu}^\alpha = b_{i\mu}^\alpha(x; \psi^\alpha, \psi^\alpha_{,\mu})$ according to (12), the GNI corresponding to (14) are

$$a_i^\alpha \frac{\delta \mathscr{L}}{\delta \psi^\alpha} - \partial_\mu \left(b_{i\mu}^\alpha \frac{\delta \mathscr{L}}{\delta \psi^\alpha} \right) - \tilde{U}_i(1) = 0 \tag{15}$$

Under the transformation (14), if the Lagrangian $\mathscr{L}$ is invariant (up to a divergence term), then the GNI (15) reduce to the classical Noether identities.

3. STRONG CONSERVATION LAWS

According to the GNI, for a certain case, we can obtain strong conservation laws, i.e. conservation laws valid whether the equations of motion are satisfied or not.

Consider the infinitesimal transformation

$$\begin{aligned} x_\mu \to x'_\mu &= x_\mu + c_{i\mu} \mathscr{E}^i \\ \psi^\alpha \to \psi^{\alpha'} &= \psi^\alpha + a_i^\alpha \mathscr{E}^i + b_{i\mu}^\alpha \, \partial_\mu \mathscr{E}^i \end{aligned} \tag{16}$$

where $c_{i\mu}$, a_i^α, $b_{i\mu}^\alpha$ are functions of x, ψ^α, $\psi^\alpha_{,\mu(m)}$. Suppose the change of the Lagrangian $\mathscr{L}$ is given by (6) and (7). From the expression (8) one obtains the identity

$$\frac{\delta \mathscr{L}}{\delta \psi^\alpha} (S_i^\alpha - \psi^\alpha_{,\mu} c_{i\mu}) \mathscr{E}^i + \partial_\mu j_\mu = U_i \mathscr{E}^i \tag{17}$$

where

$$j_\mu = \sum_{m=0}^{N-1} \Pi_\alpha^{\mu\nu(m)} \partial_{\nu(m)} (S_i^\alpha - \psi^\alpha_{,\rho} c_{i\rho}) \mathscr{E}^i + \mathscr{L} \delta_{\mu\nu} c_{i\nu} \mathscr{E}^i \tag{18}$$

multiplying the GNI (12) by $\mathscr{E}^i$ and subtracting from (17) gives us the strong conservation law

$$\partial_\mu J_\mu = 0 \tag{19}$$

$$\begin{aligned} J_\mu = j_\mu + b_{i\mu}^\alpha \frac{\delta \mathscr{L}}{\delta \psi^\alpha} \mathscr{E}^i - [u_i^{\mu(m)} \partial_{\nu\lambda\cdots\sigma\rho} \mathscr{E}^i - \delta_{\mu\nu} (\partial_\tau u_i^{\tau(m)}) \, \partial_{\lambda\cdots\sigma\rho} \mathscr{E}^i \\ + \cdots + (-1)^{2m-1} \delta_{\mu\rho} (\partial_{\tau\nu\lambda\cdots\sigma} u_i^{\tau(m)}) \mathscr{E}^i] \end{aligned} \tag{20}$$

where

$$u_i^{\mu(m)} = u_i^{\overbrace{\mu\nu\lambda\cdots\sigma\rho}^{m}}, \qquad \partial_{\mu(m)} = \partial_{\underbrace{\mu\nu\lambda\cdots\sigma\rho}_{m}} \tag{21}$$

In non-Abelian gauge theory, the Lagrangian without ghosts violates unitarity and hence the effective Lagrangian is

$$\mathscr{L}_{\text{eff}} = -\tfrac{1}{4}F_{a\mu\nu}F^{a}_{\mu\nu} + \frac{1}{2\alpha}(\partial_\mu A^a_\mu)^2 - \partial_\mu \eta^\dagger_a D^a_{b\mu}\eta^b \tag{22}$$

where A^a_μ are YM fields, $F^a_{\mu\nu}$ are field strengths, and $\eta^\dagger_a$, η^b are the ghost fields. As is well known, the Lagrangian (22) is invariant under the BRS transformation. But if we consider only the transformation of the gauge fields, fixing the ghost fields, then

$$\delta A^a_\mu = D^a_{b\mu}\theta^b, \qquad \delta\eta^\dagger_a = 0, \qquad \delta\eta^b = 0 \tag{23}$$

where $\theta^a = -\xi\eta^a$, and $D^a_{b\mu}$ represents the covariant derivative

$$D^a_{b\mu} = \delta^a_b\partial_\mu - f^a_{bc}A^c_\mu \tag{24}$$

Under the transformation (23), the effective Lagrangian (22) is variant. According to (19), we obtain the strong conservation law

$$\partial_\mu J_\mu = 0 \tag{25}$$

where

$$\begin{aligned} J_\mu = j_\mu &+ \frac{\delta\mathscr{L}_{\text{eff}}}{\delta A^a_\mu}\theta^a - \frac{1}{\alpha}[(\partial_\nu A^a_\nu)\,\partial_\mu\theta_a - \partial_\mu(\partial_\nu A^a_\nu)\theta_a] \\ &+ f^a_{bc}\left[\frac{1}{\alpha}(\partial_\nu A^b_\nu)A^c_\mu\theta_a + \partial_\mu\eta^{c+}\eta^b\theta_a\right] \end{aligned} \tag{26}$$

$$j_\mu = -\left[F_{a\mu\nu} + \frac{1}{\alpha}\delta_{\mu\nu}(\partial_\lambda A_{a\lambda})\right]D^a_{b\nu}\theta^b \tag{27}$$

Similarly, if we fix the gauge fields A^a_μ and change only the ghost fields $\eta^\dagger_a$ (or η^b), the strong conservation law (19) implies a trivial identity.

4. THE MASSIVE YANG–MILLS FIELDS

We give some preliminary applications of GNI to the massive YM field theory.

Consider a Lagrangian of the massive YM fields (Hsu and Sudarshan, 1974; Zhao and Yan, 1978)

$$\mathscr{L} = -\tfrac{1}{4}F_{a\mu\nu}F^a_{\mu\nu} + \tfrac{1}{2}m^2(A^a_\mu)^2 + \chi_a\partial_\mu A^a_\mu + \tfrac{1}{2}\alpha(\chi^a)^2 \tag{28}$$

where $\chi_a(x)$ are the multiplier fields. Under the gauge transformation

$$\delta A^a_\mu = D^a_{b\mu}\mathscr{E}^b(x) \tag{29}$$

where $\mathscr{E}^b(x)$ are arbitrary functions, the GNI (15) can be converted to

$$D_\mu D_\nu F^a_{\mu\nu} = 0 \tag{30}$$

Along the trajectory of the motion

$$D^a_{b\nu} F^b_{\nu\mu} + m^2 A^a_\mu - \partial_\mu \chi^a = 0 \tag{31}$$

$$\partial_\mu A^a_\mu + \alpha \chi^a = 0 \tag{32}$$

the GNI (15) can be reduced to

$$\partial_\mu (D^a_{b\nu} F^b_{\nu\mu} + m^2 A^a_\mu - \partial_\mu \chi^a) = m^2 \partial_\mu A^a_\mu - D^a_{b\mu} \partial_\mu \chi^b \tag{33}$$

Since $\partial_\mu \partial_\nu F^a_{\mu\nu} = 0$, one obtains (weak) conservation laws

$$\partial_\mu J^a_\mu = 0 \tag{34}$$

$$J_\mu = f^a_{bc}(A^b_\nu F^c_{\mu\nu} + A^b_\mu \chi^c) \tag{35}$$

For pure massive YM fields ($\chi_a = 0$), along the trajectory in this circumstance the GNI become conserved current equations but do not give the Lorentz condition as Iosif'yan and Konopleva (1971) say.

Similarly, the Lagrangian

$$\mathscr{L} = -\tfrac{1}{4} F_{a\mu\nu} F^a_{\mu\nu} + \tfrac{1}{2} m^2 (A^a_\mu)^2 - \bar{\psi} \gamma_\mu (\partial_\mu - i T^a A^a_\mu) \psi \tag{36}$$

is variant under the local gauge transformation

$$\begin{aligned} \psi \to \psi' &= \psi + i T^a \mathscr{E}^a(x) \psi \\ A^a_\nu \to A^{a'}_\mu &= A^a_\mu + D^a_{b\mu} \mathscr{E}^b(x) \end{aligned} \tag{37}$$

where T^a are the generators of the gauge group. Using the GNI, along the trajectory we obtain

$$\partial_\mu J^a_\mu = 0 \tag{38}$$

$$J^a_\mu = i \bar{\psi} \gamma_\mu T^a \psi + f^a_{bc} A^b_\nu F^c_{\mu\nu} \tag{39}$$

Under the transformation of the Fermi fields

$$\psi \to \psi + i\mathscr{E}(x) \gamma_5 \psi, \qquad \bar{\psi} \to \bar{\psi} + \bar{\psi} i \gamma_5 \mathscr{E}(x) \tag{40}$$

the change of the Lagrangian (36) is

$$\delta \mathscr{L} = -i \mathscr{E}(x)\, \partial_\mu j^5_\mu, \qquad j^5_\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi \tag{41}$$

According to the GNI along the trajectory of the motion one obtains conservation of the axial current, $\partial_\mu j^5_\mu = 0$.

From the above example, we see that the GNI may be converted into the equations of current conservation even if the Lagrangian of the system is not invariant under the specific transformation. This differs from the usual result (first Noether theorem), where invariance under a finite continuous group implies the conservation laws.

5. DIRAC'S CONSTRAINT

As is well known, if the Lagrangian of the system is gauge-invariant, then the system has a Dirac constraint. For variant Lagrangians we can further examine the Dirac constraint of the system using the GNI.

If the Lagrangian $\mathscr{L}$ involves no higher derivative of ψ^α than the first order, under the transformation

$$\delta x_\mu = 0, \qquad \delta\psi^\alpha = a_i^\alpha \mathscr{E}^i + b_{i\mu}^\alpha \partial_\mu \mathscr{E}^i \tag{42}$$

where a_i^α, $b_{i\mu}^\alpha$ are the functions of x, ψ^α, suppose the change of the Lagrangian $\mathscr{L}$ is

$$\delta\mathscr{L} = U_i \mathscr{E}^i = (u_i + u_i^\mu \partial_\mu + u_i^{\mu\nu} \partial_\mu \partial_\nu)\mathscr{E}^i \tag{43}$$

where u_i, u_i^μ, $u_i^{\mu\nu}$ are functions of x, ψ^α, $\psi^\alpha_{,\rho}$. According to the GNI (12), one obtains the identities (15). If $u_i^{\mu\nu}$ do not involve the derivative of ψ^α, then $\tilde{U}_i(1)$ do not involve the third-order derivative of ψ^α. When the expressions $\delta\mathscr{L}/\delta\psi^\alpha$ are substituted into the identities (15), this leads to terms containing third derivatives of ψ^α, and these must cancel each other irrespective of other terms (Bergmann, 1949):

$$b_{i\lambda}^\alpha H_{\sigma\rho}^{\alpha\beta} \psi^\beta_{,\lambda\sigma\rho} = 0 \tag{44}$$

where

$$H_{\sigma\rho}^{\alpha\beta} = \partial^2\mathscr{L}/\partial\psi^\alpha_{,\sigma}\,\partial\psi^\beta_{,\rho} \tag{45}$$

The conditions (44) are to be fulfilled for any third derivative of ψ^α, and one obtains

$$b_{i\lambda}^\alpha H_{\sigma\rho(\lambda\sigma\rho)}^{\alpha\beta} = 0 \tag{46}$$

where $(\lambda\sigma\rho)$ indicate that the expressions are symmetrized with respect to these indices. Letting $\lambda = \sigma = \rho$, we get

$$b_{i4}^\alpha H_{44}^{\alpha\beta} = 0 \tag{47}$$

For the gauge transformation, if the b_{i4}^α are not all identically zero, which implies $\det|H_{44}^{\alpha\beta}| = 0$, then the Hessian of $\mathscr{L}$ is singular, and therefore the system has a Dirac constraint.

Suppose the Lagrangian $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1$, where $\mathscr{L}_0$ is invariant (up to a divergence term) and $\mathscr{L}_1$ is variant under such a transformation. If $\mathscr{L}_1$ does not involve a derivative of ψ^α or at most involves first-order terms of $\psi^\alpha_{,\rho}$, then obviously $\tilde{U}_i(1)$ do not involve $\psi^\alpha_{,\lambda\sigma\rho}$. For example, the Lagrangian of a pure massive YM field has the properties, and has a Dirac constraint. Hence, for some noninvariant systems, the GNI can give criterion for the system to have a Dirac constraint.

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